TRASFORMATA DI LAPLACE

Esercizi proposti

1. Determinare le trasformate di Laplace delle seguenti funzioni:

   (a) $(t^2+1)^2$ $\left[\dfrac{24+4s^2+s^4}{s^5}\right]$

   (b) $e^{-t}\cos 2t$ $\left[\dfrac{s+1}{s^2+2s+5}\right]$

   (c) t^3e^{-3t} $\left[\dfrac{6}{(s+3)^4}\right]$

   (d) $\cosh 5t + \frac{1}{5}\sinh 5t$ $\left[\dfrac{s+1}{s^2-25}\right]$

   (e) $\frac{1}{2}(t+2)^2e^t$ $\left[\dfrac{2s^2-2s+1}{(s-1)^3}\right]$

   (f) $e^{-t/2}(\sin t)^2$ $\left[\dfrac{16}{(2s{+}1)(4s^2{+}4s{+}17)}\right]$

2. Dalla trasformata di $\sin t$, dedurre le trasformate delle seguenti funzioni:

   (a) $\sin(-3t)$ $\left[\dfrac{-3}{s^2+9}\right]$

   (b) $\dfrac{\sin 2t}{t}$ $\left[\dfrac{\pi}{2}-\arctan\dfrac{s}{2}=\arctan\dfrac{2}{s}\right]$

   (c) $\dfrac{d}{dt}\left(\dfrac{\sin 2t}{t}\right)$ $\left[s\arctan\dfrac{2}{s}-2\right]$

   (d) $F(t)=\displaystyle\int_0^t \frac{\sin 2u}{u}\,du$ $\left[\dfrac{1}{s}\arctan\dfrac{2}{s}\right]$

   (e) $\displaystyle\int_0^t \frac{\sin 2u}{u\,e^{3t}}\,du$ $\left[\dfrac{1}{s}\arctan\dfrac{2}{s{+}3}\right]$

3. Determinare le trasformate di Laplace delle seguenti funzioni:

   (a) $\begin{cases} 0, & 0<t<1 \\ (t-1)^2, & 1\le t \end{cases}$ $\left[\dfrac{2e^{-s}}{s^3}\right]$

   (b) $\begin{cases} 1+\cos t, & 0<t<2\pi \\ \cos t, & 2\pi\le t \end{cases}$ $\left[\dfrac{s}{s^2+1}+\dfrac{1-e^{-2\pi s}}{s}\right]$

   (c) $F(t)=t$ per $0<t\le 2\pi$, e F 2π-periodica $\left[\dfrac{e^{2\pi s}-2\pi s-1}{s^2(e^{2\pi s}-1)}\right]$

   (d) $F(t)=t$ per $-\pi<t\le\pi$, e F 2π-periodica $\left[\dfrac{e^{2\pi s}-2\pi s e^{\pi s}-1}{s^2(e^{2\pi s}-1)}\right]$

4. Calcolare i seguenti integrali impropri:

   (a) $\displaystyle\int_0^\infty \frac{e^{-2t}\sin 2t}{t}\,dt$ $\left[\dfrac{\pi}{4}\right]$

   (b) $\displaystyle\int_0^\infty \frac{e^{-t}-e^{-2t}}{t}\,dt$ $[\ln 2]$

   (c) $\displaystyle\int_0^\infty \frac{e^{-t}(\sin t)^2}{t}\,dt$ $\left[\dfrac{\ln 5}{4}\right]$

5. Determinare un'antitrasformata di Laplace di ciascuno di:

(a) $\dfrac{1}{s^2+9}$ $\quad\left[\frac{1}{3}\sin 3t\right]$

(b) $\dfrac{3s-2}{s^2-4s+20}$ $\quad\left[e^{2t}(3\cos 4t+\sin 4t)\right]$

(c) $\dfrac{s^2-2}{(s+1)(s-2)(s-3)}$ $\quad\left[\frac{1}{12}(-e^{-t}-8e^{2t}+21e^{3t})\right]$

(d) $\dfrac{s^2}{s^2+1}$ $\quad[\delta(t)-\sin t]$

(e) $\dfrac{e^{-\pi s}}{s+2}$ $\quad\left[e^{-2(t-\pi)}\,\mathcal{U}(t-\pi)\right]$

6. Trovare la trasformata di Laplace della soluzione di ciascuno dei seguenti problemi:

(a) $\begin{cases} x''-2x'-8x=e^{4t} \\ x(0)=0,\ x'(0)=0 \end{cases}$ $\quad\left[\dfrac{1}{(s+2)(s-4)^2}\right]$

(b) $\begin{cases} tx'-3x+1=0 \\ x(0)=\frac{1}{3} \end{cases}$ $\quad\left[\dfrac{1}{3s}+\dfrac{c}{s^4}\right]$

(c) $\begin{cases} x''+x=\cos t \\ x(0)=0,\ x(\frac{\pi}{2})=\frac{\pi}{4} \end{cases}$ $\quad\left[\dfrac{s}{(s^2+1)^2}+\dfrac{x'(0)}{s^2+1}\right]$

(d) $\begin{cases} tx''+x'+tx=0 \\ x(0)=1,\ x'(0)=0 \end{cases}$ $\quad\left[\dfrac{c}{\sqrt{s^2+1}}\right]$

(e) $\begin{cases} x'''+3x''+3x'+x=0 \\ x''(0)=1,\ x'(0)=0,\ x(0)=0 \end{cases}$ $\quad\left[\dfrac{1}{(s+1)^3}\right]$

7. Usando $\mathcal{L}^{-1}$, risolvere i problemi 6 (a),(c),(e). $\quad\left[\frac{1}{36}(e^{-2t}-e^{4t}+6te^{4t}),\quad \frac{1}{2}t\sin t,\quad \frac{1}{2}t^2e^{-t}\right]$

8. Usare la trasformata di Laplace, risolvere i seguenti sistemi:

(a) $\begin{cases} x_1'=x_1+2x_2 \\ x_2'=4x_1+3x_2 \end{cases}$ con $\begin{cases} x_1(0)=3 \\ x_2(0)=0 \end{cases}$ $\quad\begin{bmatrix} e^{5t}+2e^{-t} \\ 2e^{5t}-2e^{-t} \end{bmatrix}$

(b) $\begin{cases} x_1'=2x_1-4x_2+e^t \\ x_2'=x_1-2x_2 \end{cases}$ con $\begin{cases} x_1(0)=0 \\ x_2(0)=1 \end{cases}$ $\quad\begin{bmatrix} 3(e^t-2t-1) \\ e^t-3t \end{bmatrix}$

(c) $\begin{cases} x_1'=x_1+x_2-x_3 \\ x_2'=x_1+2x_2 \\ x_3'=2x_1+3x_2 \end{cases}$ con $\begin{cases} x_1(0)=2 \\ x_2(0)=0 \\ x_3(0)=0 \end{cases}$ $\quad\begin{bmatrix} (2-t^2)e^t \\ (2t+t^2)e^t \\ (4t+t^2)e^t \end{bmatrix}$

9. Sviluppando le funzioni in serie e calcolando le trasformate termine per termine, dimostrare che

(a) $\mathcal{L}\left[\dfrac{1-\cos t}{t}\right]=-\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{2n\,s^{2n}}$

(b) $\mathcal{L}\left[\sin(t^2)\right]=-\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(4n-2)!}{(2n-1)!\,s^{4n-1}}$

10. Sia $F(t)=\begin{cases} t, & t<a \\ 0, & a\le t, \end{cases}$ con $a>0$ una costante.

(a) Determinare $f(s)=\mathcal{L}[F(t)](s)$. $\quad\left[\dfrac{1-e^{-as}(1+as)}{s^2}\right]$

(b) Determinare $g(s)=\mathcal{L}[F'(t)](s)$. $\quad\left[\dfrac{1-e^{-as}}{s}\right]$

(c) Verificare che $\quad g(s)=sf(s)-F(0)+e^{-as}\left(F(a-)-F(a+)\right)$.